On Derivation of Euler-Lagrange Equations for incompressible energy-minimizers

Citation for published version:
Karakhanyan, A & Chaudhuri, N 2009, 'On Derivation of Euler-Lagrange Equations for incompressible energy-minimizers', *Calculus of Variations and Partial Differential Equations*, vol. 36, no. 4, pp. 627-645. https://doi.org/10.1007/s00526-009-0248-z

Digital Object Identifier (DOI):
10.1007/s00526-009-0248-z

Published In:
Calculus of Variations and Partial Differential Equations

Karakhanyan, A & Chaudhuri, N 2009, 'On Derivation of Euler-Lagrange Equations for incompressible energy-minimizers' Calculus of Variations and Partial Differntial Equations, vol 36, no. 4, pp. 627-645.

ON DERIVATION OF EULER-LAGRANGE EQUATIONS FOR AREA-PRESERVING ENERGY-MINIMIZERS

NIRMALENDU CHAUDHURI AND ARAM L. KARAKHANYAN

ABSTRACT. Derivation of the system of Euler-Lagrange equations for volume-preserving, energy-minimizing $W^{1,2}$-deformations and establishing the existence of an integrable pressure associated with the volume constraint is an open problem. In this article we consider this problem for the case $n = 2$. For an area-preserving, elastic energy-minimizing deformation $\mathbf{u}$ with $|\nabla \mathbf{u}|^2$ in the Hardy space $\mathcal{H}^1$, we establish an explicit representation of the associated pressure $p \in L^1_{\mathrm{loc}}$ via Calderón-Zygmund type singular integral operators. We then derive the system of Euler-Lagrange equations for $W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2), r \geq 3$ area-preserving local minimizers and prove partial regularity under smallness assumption on pressure.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth, bounded and simply connected domain. The classical Stokes problem in hydrodynamics involves minimizing the potential energy

$$I[\mathbf{w}] := \int_\Omega \frac{1}{2}|\nabla \mathbf{w}|^2 + \langle \mathbf{f}, \mathbf{w} \rangle$$

for all divergence free velocity fields $\mathbf{w} \in W_0^{1,2}(\Omega, \mathbb{R}^n)$ for a given force field $\mathbf{f} \in L^2(\Omega, \mathbb{R}^n)$. It follows that the problem has a unique incompressible minimizer $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^n)$. The linear incompressible constraint $\operatorname{div} \mathbf{u} = 0$ ensures the existence of a hydrostatic pressure $p \in L^2_{\mathrm{loc}}(\Omega)$ and the pair $(\mathbf{u}, p)$ satisfies the following system of Euler-Lagrange equations

$$\begin{cases} \Delta \mathbf{u}(x) = \nabla p(x) - \mathbf{f}(x), & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

in the weak sense, see for example [Ev 98, pp 472-474]. The regularity of $(\mathbf{u}, p)$ is well understood and detailed analysis can be found in [Ga 94, Chapter IV].

An analogue of this problem appears in nonlinear elasticity. In such context, $\mathbf{w}$ represents the displacement of an incompressible elastic body which has the rest configuration $\Omega \subset \mathbb{R}^n$. For incompressible neo-Hookean materials [Ba 77], [TO 81], [Og 84], such as vulcanized rubber, in the equilibrium state, one is interested in

---

Date: December 18, 2007.
2000 *Mathematics Subject Classification.* Primary 35J60, 73C50, 73V25.
Key words: Area-preserving maps, Calderón-Zygmund operator, convexity, elliptic estimate, energy-minimizers, Green's function and Laplace equation,

minimizing the elastic energy

$$E[\mathbf{w}] := \int_\Omega L(\nabla \mathbf{w}(x))dx\,, \tag{1.2}$$

for incompressible $W^{1,2}$-deformations $\mathbf{w} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, subject to its own boundary condition and corresponding to a given bulk energy $L : \mathbb{M}^{n\times n} \to \mathbb{R}$. The simplest L is the Dirichlet energy, given by $L(X) = \frac{1}{2}|X|^2 := \frac{1}{2}\mathrm{tr}\,(X^t X)$. Let us denote the admissible set of deformations

$$\mathcal{A} := \left\{ \mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^n) : \mathrm{cof}\, \nabla \mathbf{w} \in L^2(\Omega, \mathbb{M}^{n\times n}),\, \det \nabla \mathbf{w} = \mathbf{1} \text{ a.e.} \right\}, \tag{1.3}$$

where $W^{k,p}$ denotes the usual *Sobolev spaces* [Ad 75] and cof P is the *cofactor* matrix, whose ij-th entries is the determinant of $(n-1)\times(n-1)$ submatrix obtained by deleting the i-th row and the j-th column from the $n \times n$ matrix P . We call $\mathbf{u} \in \mathcal{A}$ to be a *local minimizer* of $E[\cdot]$ if and only if

$$E[\mathbf{u}] \le E[\mathbf{w}] \quad \text{for all } \mathbf{w} \in \mathcal{A} \text{ and } \mathrm{supp}\,(\mathbf{w} - \mathbf{u}) \subset \Omega\,. \tag{1.4}$$

Under the hypothesis that the energy density L is quasiconvex [Mo 52] and have quadratic growth, using direct methods in the calculus of variations together with weak continuity of determinant, Ball [Ba 77] proved the existence of local minimizers $\mathbf{u} \in \mathcal{A}$ of the energy $E[\cdot]$. However the derivation of the system of Euler-Lagrange equations for such minimizers and proving the existence of an integrable pressure associated with the volume constraint is a challenging open problem.

We will be concerned in this paper with the derivation of Euler-Lagrange equations for the area-preserving local minimizers and the existence of a locally integrable pressure in the planar case $n = 2$. Our main results are as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected and bounded domain. Assume that $\boldsymbol{u} \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathcal{A} = \{\boldsymbol{w} \in W^{1,2}(\Omega, \mathbb{R}^2) \,:\, \det\nabla \boldsymbol{w}(x) = 1, \text{a.e. in } \Omega\}$, for some $r \ge 3$ is a local minimizer of $E[\cdot]$. Then there exists a scalar function $q \in L^{r/2}_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$ such that the pair $(\boldsymbol{u}, p)$ satisfies*

$$\int_\Omega DL(\nabla \boldsymbol{u}(x)) \,:\, \nabla\phi(x)\, dx = \int_\Omega p(x)\, \mathrm{cof}\,(\nabla \boldsymbol{u}(x)) \,:\, \nabla \phi(x)\, dx\,, \tag{1.5}$$

for all $\phi \in C^1_0(\Omega, \mathbb{R}^2)$, where $p := q \circ \boldsymbol{u} \in L^{r/2}_{\mathrm{loc}}(\Omega)$ and $A : B := \sum_{ij} a_{ij} b_{ij}$, for $A, B \in \mathbb{M}^{2\times 2}$. In other words, the pair $(\boldsymbol{u}, p)$ satisfies the system of Euler-Lagrange equations

$$\mathrm{div}\, [DL(\nabla \boldsymbol{u}(x)) - p(x)\, \mathrm{cof}\,(\nabla \boldsymbol{u}(x))] = 0 \qquad \text{in } \Omega, \tag{1.6}$$

in the sense of distribution, where the divergence is taken in each rows.

Under the stronger hypothesis that the local minimizers of $E[\cdot]$ are classical, namely in Sobolev spaces $W^{2,r}, r > 2$, Tallec and Oden [TO 81] established the above system of equations. Whereas, our approach to establish the existence of a pressure $p \in L^{r/2}$ associated with the local minimizer $\mathbf{u}$, we only require $\mathbf{u} \in W^{1,r}_{\mathrm{loc}}$, $r > 2$ and to derive the system of equilibrium equations (1.6) for $(\mathbf{u}, p)$ in Ω we need $r \ge 3$.

Recall that for $f \in L^1(\mathbb{R}^n)$ the maximal function Mf is defined by

$$(Mf)(x) := \sup_{\rho>0} \frac{1}{\mathrm{meas}\, B_\rho(x)} \int_{B_\rho(x)} |f(y)|\, dy\,.$$

From the classical results in singular integrals due to Stein [St 69, Theorem 1] or [St 70, pp 23], it follows that if $f \in L^1(\mathbb{R}^n)$ and is supported on a finite ball $B \subset \mathbb{R}^n$, then $Mf \in L^1(B)$ is and only if

$$\begin{aligned} f \in L\log L &:= \left\{ g : B \to \mathbb{R} \, : \, \int_B |g| \log^+ |g| \, dx < \infty \right\} \\ &\equiv \left\{ g : B \to \mathbb{R} \, : \, \int_B |g| \log(2 + |g|) \, dx < \infty \right\} , \end{aligned}$$

where $\log^+ |x| = 0$ for $0 < |x| \leq 1$ and $\log^+ |x| = \log |x|$ for $|x| > 1$. A standard result states that a positive function f is in the *Hardy space* $\mathcal{H}^1$ (the pre dual of BMO) if and only if $f \in L \log^+ L$. Notice that without any further higher integrability assumption on $\nabla \mathbf{u}$, we cannot ensure integrability of the maximal function $M|\nabla \mathbf{u}|^2$. However, under the additional assumption that $M|\nabla \mathbf{u}|^2$ is integrable, which is equivalent to $|\nabla \mathbf{u}|^2 \in \mathcal{H}^1$, we prove that the pressure q on the deformed domain $\mathbf{u}(\Omega)$ is locally integrable and $(\mathbf{u}, q)$ satisfies the same system of differential equations a very weak sense. More precisely, we prove the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected, bounded domain. Assume that $\boldsymbol{u} \in \mathcal{A}$ is a local minimizer of $E[\cdot]$ such that $|\nabla \boldsymbol{u}|^2 \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$. Then there exists $q \in L^1_{loc}(\boldsymbol{u}(\Omega))$ such that the pair $(\boldsymbol{u}, q)$ satisfies the integral identity*

$$\int_\Omega DL(\nabla \boldsymbol{u}(x)) \, : \, \nabla(\boldsymbol{v} \circ \boldsymbol{u}) \, dx = \int_{\boldsymbol{u}(\Omega)} q(z) \, \operatorname{div} \boldsymbol{v}(z) \, dz \tag{1.7}$$

for all $\boldsymbol{v} \in C_0^\infty(\boldsymbol{u}(\Omega), \mathbb{R}^2)$.

The proof of Theorem 1.2 is quite delicate. The main ideas in our proof are to localize the *mollified pressure* on the deformed domain $\mathbf{u}(\Omega)$, its explicit representation using Green's function of the unit disc in $\mathbb{R}^2$ and finding its uniform bound by using Calderón-Zygmund estimate [CZ 52]. Finally we show that the pressure on $\mathbf{u}(\Omega)$ is locally represented as the sum of certain singular integral operators of $|\nabla \mathbf{u}|^2$ involving Calderón-Zygmund type kernels (see equation (4.17) in Section 4) [CZ 52].

Theorem 1.3. [CZ 52, **Calderón-Zygmund Theorem**] *Let $f \in L \log^+ L$ and let Γ be a C^1 function on $\mathbb{R}^n \setminus \{0\}$ homogeneous of degree 0 with mean value 0 over the unit sphere $\mathbb{S}^{n-1}$, that is*

$$\int_{\mathbb{S}^{n-1}} \Gamma(x) \, dS(x) = 0. \tag{1.8}$$

Then the function defined as

$$f^*(x) := \lim_{\delta \to 0} \int_{|x-y| \geq \delta} \frac{\Gamma(x-y)}{|x-y|^n} f(y) \, dy \tag{1.9}$$

exists a.e. and integrable. Furthermore,

$$\int_K |f^*| \, dy \leq C \int_{\mathbb{R}^n} |f| \left(1 + \log^+ \left((\operatorname{meas} K)^{\frac{n+1}{n}} |f|\right)\right) dy + C(\operatorname{meas} K)^{-\frac{1}{n}} , \tag{1.10}$$

for all measurable subset K of $\mathbb{R}^n$ with finite measure.

For $n = 2$, through a series of papers, Bauman, Owen and Phillips [BOP 91], [BOP 91a], [BOP 92] proved that any $W^{2,r}$, $r > 2$ solutions of (1.6) are smooth solutions. In 1999, Evans and Gariepy [EG 99] proved that any *non-degenerate*, Lipschitz area-preserving local minimizers of $E[\cdot]$ are $C^{1,\alpha}(\Omega_0)$, for some $0 < \alpha < 1$ for a dense open subset $\Omega_0 \subset \Omega$. However, as a consequence of the Euler-Lagrange equations (1.6) together with the standard elliptic estimates [GM 79]we prove the following theorem.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $L : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a smooth and uniformly convex function having quadratic growth. Assume that $\boldsymbol{u} \in \mathcal{A} \cap W^{1,3}_{\text{loc}}(\Omega, \mathbb{R}^2)$ be a local minimizer of $E[\cdot]$ and $q(z) \in C^\alpha$ for some positive α. Then $\boldsymbol{u}$ has Hölder continuous first derivatives in sumbdomain Ω_0. Moreover*

$$|\Omega \setminus \Omega_0| = 0.$$

In a forthcoming paper [CHK 08] we will discuss the regularity of $W^{1,r}_{\text{loc}}$, $r > 2$- area-preserving local minimizers and the derivation of system of Euler-Lagrange equations for the case $n \geq 3$.

2. The First Variation of Energy

In the study of regularity of finite energy deformations, Šverák [Sv 88] proved that for any $W^{1,n}$-deformation $\mathbf{w}$ with $\det \nabla \mathbf{w}(x) > 0$, a.e., there exists a continuous function ω on $\mathbb{R}$ with $\omega(0) = 0$ such that

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq \omega(|x - y|), \quad \text{for any } x, y \in \Omega \subset\subset \mathbb{R}^n.$$

In connection to the study of quasi-regular maps for $n = 2$, Iwaniec and Šverák [IS 93] proved that any $W^{1,2}$-deformation $\mathbf{w}$ with the *distortion* function $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^2 / \det \nabla \mathbf{w}(\cdot)$ being integrable, $\mathbf{w}$ is a homeomorphism. Thus in particular, area-preserving $W^{1,2}$-deformations in the plane are continuous and open maps. For $n \geq 3$, it is still unknown whether a map $\mathbf{u} \in \mathcal{A}$ is a homeomorphism.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth bounded domain. Let $L : \mathbb{M}^{n\times n} \to \mathbb{R}$ be a smooth function and $\boldsymbol{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. For $n \geq 3$, we further assume that $\boldsymbol{u}$ is a continuous and an open map. Then $\boldsymbol{u}$ satisfies the following integral identity*

$$\int_\Omega DL(\nabla \boldsymbol{u}(x)) : \nabla(\boldsymbol{v} \circ \boldsymbol{u})(x)\, dx = 0\,, \tag{2.1}$$

for all smooth, compactly supported and divergence free vector fields $\boldsymbol{v}$ on $\mathbf{u}(\Omega)$, where $A : B := \mathrm{tr}(\mathrm{A^t B}) = \sum_{\mathrm{i,j=1}}^{\mathrm{n}} \mathrm{a_{ij} b_{ij}}$ is the scalar product on $\mathbb{M}^{n\times n}$.

Proof: Let $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$ be a vector field with $\operatorname{div} \mathbf{v} = 0$. For each $y \in \mathbf{u}(\Omega)$, consider the unique smooth flow $\phi(y, \cdot) : \mathbb{R} \to \mathbf{u}(\Omega)$ given by

$$\frac{d\phi}{dt}(y, t) = \mathbf{v}(\phi(y, t)) \quad \text{in} \quad \mathbb{R}, \quad \phi(y, 0) = y. \tag{2.2}$$

Using the relations $\dfrac{\partial}{\partial P_{ij}} \det P = (\operatorname{cof} P)_{ij}$ and $P\,(\operatorname{cof} P)^t = I_n \det P$, by a direct calculations we observe that

$$\frac{d}{dt}\left(\det \nabla_y \phi(y, t)\right) = \det \nabla_y \phi(y, t)\ \operatorname{div} \mathbf{v} = 0. \tag{2.3}$$

Since $\det \nabla_y \phi(y, 0) = 1$, from (2.3) it follows that $\det \nabla_y \phi(y, t) = 1$ for all $t \in \mathbb{R}$ and $y \in \mathbf{u}(\Omega)$. Consider the map $\mathbf{w} : \Omega \times \mathbb{R} \to \mathbf{u}(\Omega)$ defined by

$$\mathbf{w}(x, t) := \phi(\cdot, t) \circ \mathbf{u}\,(x) = \phi(\mathbf{u}(x), t) \quad \text{for any } t \in \mathbb{R},\ x \in \Omega.$$

Let $V := \operatorname{supp} \mathbf{v} \subset \mathbf{u}(\Omega)$, then $\mathbf{v}(\mathbf{u}(x)) = 0$ for $\mathbf{u}(x) \notin V$. This in conjunction with the uniqueness of ϕ implies that $\phi(\mathbf{u}(x), t) = \mathbf{u}(x)$ for all points x such that $\mathbf{u}(x) \notin V$. Since Ω is bounded, $\mathbf{u}$ is continuous and V is compact, $\Omega' = \mathbf{u}^{-1}(V)$ is a compact subset of Ω. Hence $\operatorname{supp}(\mathbf{w}(x,t) - \mathbf{u}(x)) \subset \Omega'$. Furthermore, $\det \nabla_x \mathbf{w}(x,t) = \det \nabla_y \phi(y,t) \det \nabla \mathbf{u}(x) = 1$. Therefore, $\mathbf{w}(\cdot, t) \in \mathcal{A}$ and $\operatorname{supp}(\mathbf{u} - \mathbf{w}(\cdot,t)) \subset \Omega$ for all $t \in \mathbb{R}$. Since $\mathbf{u}$ is a local minimizer of $E[\cdot]$,

$$E[\mathbf{u}] \leq E[\mathbf{w}(\cdot, t)] \qquad \text{for all} \quad t \in \mathbb{R}.$$

Thus in particular,

$$\begin{aligned}
0 &= \frac{d}{dt} \int_\Omega L(\nabla \mathbf{w}(x,t))\, dx \Big|_{t=0} \\
&= \sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{w}(x,t)) \frac{d}{dt} \left(\frac{\partial w^i}{\partial x_j}(x,t) \right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{w}(x,t)) \frac{\partial}{\partial x_j} \left(\frac{d\phi^i}{dt}(\mathbf{u}(x),t) \right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{w}(x)) \frac{\partial}{\partial x_j} \left(v^i(\phi(u(x),t) \right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} \left(v^i(\mathbf{u}(x)) \right) dx \\
&= \int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x)\, dx \,,
\end{aligned}$$

for all smooth, compactly supported and divergence free vector fields on $\mathbf{u}(\Omega)$, where $L_{ij}(P) := \dfrac{\partial L}{\partial P_{ij}}(P)$. This proves the Theorem. □

3. Derivation of Euler-Lagrange Equations for $n = 2$

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply connected domain. Assume that the bulk energy $L : \mathbb{M}^{2\times 2} \to \mathbb{R}$ is smooth such that, $|L(P)| \leq C(1+|P|^2)$, $|DL(P)| \leq C(1+|P|)$ and $|D^2 L(P)| \leq C$ for all $P \in \mathbb{M}^{2\times 2}$, for some $C > 0$. Since $|\operatorname{cof} P| = |P|$ for $P \in \mathbb{M}^{2\times 2}$, the area-preserving maps in the plane $\mathcal{A}$ defined in (1.3) is equivalent to the family $\{\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2) \,:\, \det \nabla \mathbf{w}(x) = 1, \text{a.e. in } \Omega\}$. Let $\mathbf{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. Then $\mathbf{u} : \Omega \to \mathbf{u}(\Omega)$ is an open map and a local homeomorphism [Sv 88], [IS 93]. Throughout this section we denote $V \subset\subset \mathbf{u}(\Omega)$, a smooth and simply connected sub-domain, C is a generic absolute constant depending only on Ω, V, and L. Its value can vary from line to line, but each line is valid with C being a pure positive number.

Let $\mathbf{v} = (v^1, v^2) \in C_0^\infty(V, \mathbb{R}^2)$ such that $\operatorname{div} \mathbf{v} = 0$. Let ρ be the usual mollification kernel. For $0 < \varepsilon < \operatorname{dist}(V, \partial\, \mathbf{u}(\Omega))$, let $\mathbf{v}_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2)$ be the mollification of $\mathbf{v}$, where

$$v_\varepsilon^i(y) := (v^i * \rho_\varepsilon)(y) = \int_{\mathbb{R}^2} \rho_\varepsilon(y-z) v^i(z) dz = \int_V \rho_\varepsilon(y-z) v^i(z) dz, \quad y \in \mathbf{u}(\Omega).$$

Thus $\mathbf{v}_\varepsilon \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^2)$ and $\operatorname{div} \mathbf{v}_\varepsilon = 0$. Hence by testing the identity (2.1) with $\mathbf{v} = \mathbf{v}_\varepsilon$, we obtain

$$\sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j}(v_\varepsilon^i \circ \mathbf{u})(x)\, dx = 0,$$

or in more explicitly

$$\sum_{i,j,k=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x)\, dx = 0. \tag{3.1}$$

From the definition of mollification, by taking $y = \mathbf{u}(x)$, for $x \in \Omega$, we obtain

$$\frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) = \int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z)\, v^i(z) dz. \tag{3.2}$$

Therefore by pluging (3.2) into (3.1) and Fubini's Theorem yields

$$\begin{aligned} 0 &= \sum_{i,j,k=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) \left(\int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z)\, v^i(z) dz \right) dx \\ &= \sum_{i,j,k=1}^{2} \int_V \left(\int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x)\, dx \right) v^i(z)\, dz \\ &= \sum_{i,j=1}^{2} \int_V \left(\int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \sum_{k=1}^{2} \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x)\, dx \right) v^i(z)\, dz \\ &= \sum_{i=1}^{2} \int_V \left[\sum_{j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} \Big(\rho_\varepsilon(\mathbf{u}(x) - z) \Big)\, dx \right] v^i(z) dz\,. \end{aligned} \tag{3.3}$$

Let us define the smooth function $g_\varepsilon^i : V \to \mathbb{R}$, for $i = 1, 2$ by

$$g_\varepsilon^i(z) := \sum_{j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} \Big(\rho_\varepsilon(\mathbf{u}(x) - z) \Big) dx. \tag{3.4}$$

Then $\mathbf{g}_\varepsilon = (g_\varepsilon^1, g_\varepsilon^2) \in C^\infty(V, \mathbb{R}^2)$ and

$$\begin{aligned} |\mathbf{g}_\varepsilon(z)| &\le \sum_{ij} \int_\Omega \left| L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \frac{\partial u^k}{\partial x_j}(x) \right| dx \\ &\le \frac{C}{\varepsilon^3} \Big((\operatorname{meas} \Omega)^{1/2} + \|\nabla \mathbf{u}\|_{L^2(\Omega)} \Big) \|\nabla \mathbf{u}\|_{L^2(\Omega)}. \end{aligned}$$

Thus combing (3.3) and (3.4) we get

$$\int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle\, dz = 0 \quad \text{for } \mathbf{v} \in C_0^\infty(V, \mathbb{R}^2) \text{ such that } \operatorname{div} \mathbf{v} = 0 \text{ in } V\,, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^2$. Let $\phi \in C_0^\infty(V)$ and define $\mathbf{v}(z) := J \nabla \phi(z)$ for $z \in V$, where J be the 90^0 planar rotation given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it follows that $\operatorname{div}\mathbf{v} = 0$ and hence by testing (3.5) with this particular choice of $\mathbf{v}$ and integrating by parts we obtain,

$$\begin{aligned} 0 &= \int_V \langle \mathbf{g}_\varepsilon(z), J\nabla\phi(z)\rangle \, dz \\ &= \int_V \left\langle J^t\mathbf{g}_\varepsilon(z), \nabla\phi(z)\right\rangle \, dz \\ &= -\int_V \phi(z)\, \operatorname{div}(J^t\mathbf{g}_\varepsilon(z))\, dz \qquad \text{for all } \phi \in C_0^\infty(V). \end{aligned}$$

Hence $\operatorname{curl}\mathbf{g}_\varepsilon := \dfrac{\partial g_\varepsilon^1}{\partial z_2} - \dfrac{\partial g_\varepsilon^2}{\partial z_1} = \operatorname{div}(J^t\mathbf{g}_\varepsilon) = 0$ in V. Since V is simply connected, there exists $q_\varepsilon \in C^\infty(V)$, such that

$$\mathbf{g}_\varepsilon(z) = -\nabla q_\varepsilon(z), \quad \text{for all } z \in V, \tag{3.6}$$

modulo translation of a constant.

Lemma 3.1. *Consider the family* $\boldsymbol{g}_\varepsilon$ *be given by (3.4). Then* $\boldsymbol{g}_\varepsilon \rightharpoonup \boldsymbol{g}$ *weakly in the dual space* $(C_0^1(V,\mathbb{R}^2))^*$.

Proof: Since ρ_ε is radially symmetric

$$\frac{\partial\rho_\varepsilon}{\partial y_k}(|y-z|) = {\rho_\varepsilon}'(|y-z|)\frac{y_k - z_k}{|y-z|} = -\frac{\partial\rho_\varepsilon}{\partial z_k}(|y-z|). \tag{3.7}$$

Therefore from the definition of g_ε^i in (3.4), we have

$$\begin{aligned} g_\varepsilon^i(z) &= -\sum_{j,k=1}^{2}\int_\Omega L_{ij}(\nabla\mathbf{u}(x))\,\frac{\partial u^k}{\partial x_j}(x)\frac{\partial\rho_\varepsilon}{\partial z_k}(\mathbf{u}(x)-z)dx \\ &= -\sum_{k=1}^{2}\int_\Omega \sigma_{ik}(x)\frac{\partial}{\partial z_k}\Big(\rho_\varepsilon(\mathbf{u}(x)-z)\Big)dx\,, \end{aligned} \tag{3.8}$$

where

$$\sigma_{ik}(x) := \sum_{j=1}^{2} L_{ij}(\nabla\mathbf{u}(x))\,\frac{\partial u^k}{\partial x_j}(x) \quad \text{for } x \in \Omega\,. \tag{3.9}$$

Since $\mathbf{u}$ is a $W^{1,2}$ area-preserving homeomorphism, $\nabla\mathbf{u}^{-1}(\mathbf{u}(x)) = (\operatorname{cof}\nabla\mathbf{u}(x))^t$. Thus it follows that $\mathbf{u}^{-1} \in W^{1,2}(\mathbf{u}(\Omega),\Omega)$. Using the structural assumptions on L in (3.9), we get

$$\int_{\mathbf{u}(\Omega)} |(\sigma_{ik}\circ\mathbf{u}^{-1})(z))|\, dz = \int_\Omega |\sigma_{ik}(x)|dx \le C\int_\Omega |\nabla\mathbf{u}(x)|^2 dx < \infty\,,$$

and hence $\tilde\sigma_{ik} := \sigma_{ik}\circ\mathbf{u}^{-1} \in L^1(\mathbf{u}(\Omega))$, for $i,k = 1,2$. Now observe that for any test function $\mathbf{v} \in C_0^\infty(V,\mathbb{R}^2)$, using Fubini, integration by parts and change of variable

$\xi = \mathbf{u}(x)$ we obtain

$$\begin{aligned}
(3.10)\qquad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz &= -\sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}(x) \left(\int_V \frac{\partial}{\partial z_j}(\rho_\varepsilon(\mathbf{u}(x)-z))\, v^i(z)\, dz \right) dx \\
&= \sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}(x) \left(\int_V \rho_\varepsilon(\mathbf{u}(x)-z)\, \frac{\partial v^i}{\partial z_j}(z)\, dz \right) dx \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z) \left(\int_\Omega \sigma_{ij}(x)\, \rho_\varepsilon(\mathbf{u}(x)-z)\ dx \right) dz \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z) \left(\int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi)\, \rho_\varepsilon(\xi - z)\ d\xi \right) dz \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z)\ (\tilde\sigma_{ij})_\varepsilon(z)\, dz\,,
\end{aligned}$$

where

$$(3.11)\qquad (\tilde\sigma_{ij})_\varepsilon(z) := ((\sigma_{ij}\circ \mathbf{u}^{-1}) * \rho_\varepsilon)(z) = \int_{u(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi))\, \rho_\varepsilon(\xi - z)\, d\xi,$$

is the usual mollification of $\sigma_{ij}\circ\mathbf{u}^{-1}$. Since $(\sigma_{ij}\circ\mathbf{u}^{-1}) * \rho_\varepsilon \to \sigma_{ij}\circ\mathbf{u}^{-1}$ in $L^1(\mathbf{u}(\Omega))$ as $\varepsilon \to 0$, by passing through the limit as $\varepsilon \to 0$ in (3.10) we conclude that

$$(3.12)\qquad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz \to \sum_{i,j=1}^{2} \int_V \sigma_{ij}(\mathbf{u}^{-1}(z))\, \frac{\partial v^i}{\partial z_j}(z)\,, dz \quad \text{as } \varepsilon \to 0$$

for all $\mathbf{v} \in C_0^\infty(V, \mathbb{R}^2)$. Now let us define the functional $\mathbf{g} : C_0^1(V,\mathbb{R}^2) \to \mathbb{R}$ as

$$(3.13)\qquad \langle \mathbf{g}, \mathbf{v}\rangle := \lim_{\varepsilon\to 0} \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz = \int_V \sigma(\mathbf{u}^{-1}(z)) : \nabla\mathbf{v}(z)\, dz\,,$$

for $\mathbf{v} \in C_0^1(V,\mathbb{R}^2)$, where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2\times 2}$. Then from (3.13) it follows that

$$(3.14)\qquad |\langle \mathbf{g}, \mathbf{v}\rangle| \le C\|\sigma\|_{L^1(\Omega)}\, \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))} \le C\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}\, \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))},$$

for any $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega, \mathbb{R}^2)$. Hence $\mathbf{g}$ is a continuous linear functional on $C_0^1(\mathbf{u}(\Omega), \mathbb{R}^2)$. Therefore, from the definition of $\mathbf{g}_\varepsilon$ in (3.4), it follows that $\mathbf{g}_\varepsilon \rightharpoonup \mathbf{g}$ weakly in the dual space $(C_0^1(V,\mathbb{R}^2))^*$. This proves the lemma. □

Lemma 3.2. *Assume that $\boldsymbol{u} \in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2) \cap \mathcal{A}$ for some $r > 2$. Then the family q_ε defined by $-\nabla q_\varepsilon = \boldsymbol{g}_\varepsilon$ in (3.6) is uniformly bounded in $L^{r/2}_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$.*

Proof Since $\mathbf{u} \in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ for some $r > 2$, from the definition of σ_{ij} in (3.9) and the growth condition on L, it follows that for any $V \subset\subset \mathbf{u}(\Omega)$

$$(3.15)\qquad \int_V |(\sigma_{ij}\circ\mathbf{u}^{-1})(z))|^{r/2} dz = \int_{\mathbf{u}^{-1}(V)} |\sigma_{ij}(x)|^{r/2} dx \le C \int_{\mathbf{u}^{-1}(V)} |\nabla \mathbf{u}(x)|^r dx,$$

and hence $\tilde\sigma_{ij} := \sigma_{ij}\circ\mathbf{u}^{-1} \in L^{r/2}(V)$, for $i,j = 1,2$. Let $f_\varepsilon : V \to \mathbb{R}$ be defined as $f_\varepsilon(z) := q_\varepsilon(z)|q_\varepsilon(z)|^{\frac{r}{2}-2}$, $z \in V$, so that for any $1 < s < \infty$,

$$\int_V |f_\varepsilon(z)|^s\, dz = \int_V |q_\varepsilon(z)|^{s\left(\frac{r}{2}-1\right)} dz = \big\||q_\varepsilon|^{\frac{r}{2}-1}\big\|^s_{L^s(V)}\,.$$

Translating f_ε to $f_\varepsilon - \dfrac{1}{\operatorname{meas} V}\displaystyle\int_V f_\varepsilon(z)\,dz$, if necessary, so that $\displaystyle\int_V f_\varepsilon(z)\,dz = 0$. In view of this normalization, there exists a smooth vector field $\mathbf{w}_\varepsilon : V \mapsto \mathbb{R}^2$, such that

$$\begin{cases} \operatorname{div}\mathbf{w}_\varepsilon = f_\varepsilon & \text{in } V \\ \mathbf{w}_\varepsilon = 0 & \text{on } \partial V\,. \end{cases} \tag{3.16}$$

Furthermore we have the estimate

$$\|\mathbf{w}_\varepsilon\|_{W^{1,s}(V)} \le C\|f_\varepsilon\|_{L^s(V)} = C\left\||q_\varepsilon|^{\frac{r}{2}-1}\right\|_{L^s(V)}\,, \tag{3.17}$$

for $C > 0$ independent of ε, see Dacorogna-Moser [DM 90]. Then for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
\int_V |q_\varepsilon(z)|^{r/2}dz &= \int_V q_\varepsilon(z)|q_\varepsilon(z)|^{r/2\,-\,2}q_\varepsilon(z)\,dz \\
&= \int_V q_\varepsilon(z)\operatorname{div}\mathbf{w}_\varepsilon(z)dz && \text{by (3.16)} \\
&= -\int_V \langle\nabla q_\varepsilon(z), \mathbf{w}_\varepsilon(z)\rangle\;dz \\
&= \int_V \langle\mathbf{g}_\varepsilon(z), \mathbf{w}_\varepsilon(z)\rangle\,dz && \text{by (3.6)} \\
&= \sum_{i,j=1}^{2}\int_V ((\sigma_{ij}\circ\mathbf{u}^{-1}) * \rho_\varepsilon)(z)\,\frac{\partial w_\varepsilon^i}{\partial z_k}(z)dz && \text{by (3.10)} \\
&\le C\sum_{i,j=1}^{2}\left(\int_V |\sigma_{ij}(\mathbf{u}^{-1}(z))|^{r/2}dz\right)^{2/r}\left(\int_V \left|\frac{\partial w_\varepsilon^i}{\partial z_k}(z)\right|^{r/(r-2)}dz\right)^{(r-2)/r} \\
&\le C\,\left\||q_\varepsilon|^{\frac{r}{2}-1}\right\|_{L^{r/(r-2)}(V)}\sum_{i,j=1}^{2}\|\sigma_{ij}\circ\mathbf{u}^{-1}\|_{L^{r/2}(V)} && \text{by (3.17)} \\
&= C\left(\int_V |q_\varepsilon(z)|^{r/2}dz\right)^{1-2/r}\|\sigma\|_{L^{r/2}(\mathbf{u}(\Omega),\mathbb{M}^{2\times 2})} \\
&\le C\left(\int_V |q_\varepsilon(z)|^{r/2}dz\right)^{1-2/r}\|\nabla\mathbf{u}\|^2_{L^r(\Omega)}\,. && \text{by (3.15)}
\end{aligned}$$

Hence there exists a constant $C > 0$, independent of ε such that

$$\|q_\varepsilon\|_{L^{r/2}(V)} \le C\|\nabla\mathbf{u}\|^2_{L^r(\Omega)}. \tag{3.18}$$

Since $r > 2$, there exists a function $q \in L^{r/2}(V)$, such that $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$. This proves the lemma. $\square$

Proof of Theorem 1.1 Using the change of variables, recalling the definitions of $\mathbf{g}$ in (3.13), and σ_{ij} in (3.9), we obtain

$$
\begin{aligned}
\langle \mathbf{g}, \mathbf{v}\rangle &= \sum_{i,j=1}^{2} \int_V \sigma_{ij}(\mathbf{u}^{-1}(z)) \frac{\partial v^i}{\partial z_j}(z)\, dz \\
&= \sum_{i,j=1}^{2} \int_{\mathbf{u}^{-1}(V)} \sigma_{ij}(x) \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x))\, dx \\
&= \sum_{i,k=1}^{2} \int_{\mathbf{u}^{-1}(V)} L_{ik}(\nabla \mathbf{u}(x)) \left(\sum_{j=1}^{2} \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \right) dx \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}\circ\mathbf{u})(x) dx \quad \text{for } \mathbf{v} \in C_0^1(V, \mathbb{R}^2)\,.
\end{aligned} \tag{3.19}
$$

Since $\mathbf{u}^{-1} \in W^{1,r}(V, \mathbf{u}^{-1}(V))$, for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$, the composition $\phi\circ\mathbf{u}^{-1} \in W_0^{1,r}(V, \mathbb{R}^2)$. Hence there exists $\mathbf{v}_\delta \in C_0^1(V, \mathbb{R}^2)$ such that $\mathbf{v}_\delta \to \psi := \phi\circ\mathbf{u}^{-1}$ strongly in $W^{1,r}(V, \mathbb{R}^2)$ as $\delta \to 0$. Then Hölder inequality yields

$$
\begin{aligned}
&\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \Big(\nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) \;-\; \nabla(\psi\circ\mathbf{u})(x) \Big) dx \\
&\qquad = \int_{\mathbf{u}^{-1}(V)} (\nabla \mathbf{u}(x))^t DL(\nabla \mathbf{u}(x)) : \Big(\nabla_z \mathbf{v}_\delta(\mathbf{u}(x)) - \nabla_z \psi(\mathbf{u}(x)) \Big) dx \\
&\qquad \le C \|\nabla \mathbf{u}\|_{L^{2r'}(\mathbf{u}^{-1}(V))} \, \|\nabla(\mathbf{v}_\delta - \psi)\|_{L^r(V)},
\end{aligned}
$$

where $r' = r/(r-1)$. Notice that $r \ge 3$ yields $2r' \le r$ and hence $\nabla \mathbf{u} \in L^r_{\mathrm{loc}}(\Omega) \subseteq L^{2r'}_{\mathrm{loc}}(\Omega)$. Therefore, from (3.19) we obtain

$$
\begin{aligned}
\langle \mathbf{g}, \mathbf{v}_\delta\rangle &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\delta\circ\mathbf{u})(x)\, dx \\
&\to \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\phi \,\circ\, \mathbf{u}^{-1}\circ\mathbf{u})(x)\, dx \quad \text{as } \delta\to 0 \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx\,.
\end{aligned} \tag{3.20}
$$

Now define the linear functional $\mathbf{g}\circ\mathbf{u} : C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2) \to \mathbb{R}$ by

$$
\langle \mathbf{g}\circ\mathbf{u}, \phi\rangle := \langle \mathbf{g}, \phi \,\circ\, \mathbf{u}^{-1}\rangle = \lim_{\delta\to 0} \langle \mathbf{g}, \mathbf{v}_\delta\rangle = \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx, \tag{3.21}
$$

for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. Hence $\mathbf{g}\circ\mathbf{u}$ defines a continuous linear functional on $W_0^{1,2}(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. On the other hand, since $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$, using the definition of $\mathbf{g}$, the representation of $\mathbf{g}_\varepsilon = -\nabla q_\varepsilon$ and integration by parts we conclude that

$$\begin{aligned}
\langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \lim_{\varepsilon \to 0} \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}_\delta(z) \rangle \, dz \\
&= -\lim_{\varepsilon \to 0} \int_V \langle \nabla q_\varepsilon(z), \mathbf{v}_\delta(z) \rangle \, dz \\
&= \lim_{\varepsilon \to 0} \int_V q_\varepsilon(z) \operatorname{div} \mathbf{v}_\delta(z) \, dz \\
&= \int_V q(z) \operatorname{div} \mathbf{v}_\delta(z) \, dz \\
&= \int_V q(z) \ \operatorname{tr} (\nabla_z \mathbf{v}_\delta(z)) \ dz \, .
\end{aligned} \tag{3.22}$$

The area constraint $\det \nabla \mathbf{u}(x) = 1$ a.e., and $\nabla(\mathbf{v} \circ \mathbf{u})(x) = \nabla_z \mathbf{v}(\mathbf{u}(x)) \ \nabla \mathbf{u}(x)$, yields $\nabla_z \mathbf{v}(\mathbf{u}(x)) = \nabla(\mathbf{v} \circ \mathbf{u})(x) \, (\operatorname{cof} \nabla \mathbf{u}(x))^t$. Using $\mathbf{u} \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ together with the fact that $|\operatorname{cof} P| = |P|$ for any $P \in \mathbb{M}^{2\times 2}$, we conclude that $\operatorname{cof} \nabla \mathbf{u} \in L^r_{\mathrm{loc}}(\Omega, \mathbb{M}^{2\times 2})$. Since $q \in L^{r/2}(V)$ and $L^{r/2}_{\mathrm{loc}} \subseteq L^{r/(r-1)}_{\mathrm{loc}}$ for $r \geq 3$, applying change of variables in (3.22), we obtain

$$\begin{aligned}
\langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{tr}\Big(\nabla_z \mathbf{v}_\delta(\mathbf{u}(x)) \Big) \, dx \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{tr}\Big(\nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) \ (\operatorname{cof} \nabla \mathbf{u}(x))^t \Big) \, dx \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{cof} (\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) \, dx, \\
&\to \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{cof} (\nabla \mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x) \, dx \quad \text{as } \delta \to 0 \\
&= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{cof} (\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx \, .
\end{aligned} \tag{3.23}$$

Hence from (3.21) and (3.23) we obtain

$$\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx = \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \ \operatorname{cof} (\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx \, ,$$

for any $\phi \in C^1_0(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. Finally choose a sequence of smooth, simply connected sets $V_k \subset\subset V_{k+1} \subset\subset \mathbf{u}(\Omega)$ sub-domains such that $\mathbf{u}(\Omega) = \cup_{k=1}^\infty V_k$. Utilizing the foregoing arguments and lemmas 3.1-3.2, there exists $q_k \in L^{r/2}(V_k)$, $k \geq 1$ such that

$$\int_{\mathbf{u}^{-1}(V_k)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) = \int_{\mathbf{u}^{-1}(V_k)} q_k(\mathbf{u}(x)) \ \operatorname{cof} (\nabla \mathbf{u}(x)) : \nabla \phi(x) \, , \tag{3.24}$$

for $\phi \in C^1_0(\mathbf{u}^{-1}(V_k), \mathbb{R}^2)$. Since $\mathbf{u}$ is locally area-preserving homeomorphism, $\Omega = \cup_{k=1}^\infty \mathbf{u}^{-1}(V_k)$ is an open covering of Ω and $\mathbf{u}^{-1}(V_k) \subset\subset \mathbf{u}^{-1}(V_{k+1})$. Using the identity $\operatorname{div} \operatorname{cof} \nabla \mathbf{u}(x) = 0$ and invertibility of $\nabla \mathbf{u}(x)$, from (3.24) it follows that q_k is unique up to a translation of a constant. Thus adding constant terms as necessary to each q_k, we deduce from (3.24) that for each fixed $k \geq 1$

$$q_i(z) = q_k(z) \quad \text{for } z \in V_i, \quad 1 \leq i \leq k.$$

We finally define $q : \mathbf{u}(\Omega) \to \mathbb{R}$ as $q(z) := q_k(z)$, for $z \in V_k$, so that $q \in L^{r/2}_{\mathrm{loc}}(\mathbf{u}(\Omega))$. This proves that for any $\phi \in C^1_0(\Omega, \mathbb{R}^2)$, the pair $(\mathbf{u}, q)$ satisfies

$$\int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla \phi(x)\, dx = \int_\Omega q(\mathbf{u}(x))\ \mathrm{cof}\,(\nabla \mathbf{u}(x)) : \nabla \phi(x)\, dx\,.$$

Now let us define the pressure p on Ω by

$$p(x) := q(\mathbf{u}(x)) \quad \text{for } x \in \Omega.$$

Then for any $k \geq 1$,

$$\int_{\mathbf{u}^{-1}(V_k)} |p(x)|^{r/2} = \int_{\mathbf{u}^{-1}(V_k)} |q(\mathbf{u}(x))|^{r/2} dx = \int_{V_k} |q(z)|^{r/2} dz < \infty,$$

and hence $p \in L^{r/2}_{\mathrm{loc}}(\Omega)$ and the pair $(\mathbf{u}, p)$ satisfies

$$\int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla \phi(x)\, dx = \int_\Omega p(x)\ \mathrm{cof}\,(\nabla \mathbf{u}(x)) : \nabla \phi(x)\, dx\,, \tag{3.25}$$

for any $\phi \in C^1_0(\Omega, \mathbb{R}^2)$. In other words, $(\mathbf{u}, p)$ satisfies the system of Euler-Lagrange equations

$$\mathrm{div}\,[DL(\nabla \mathbf{u}(x)) - p(x)\,\mathrm{cof}\,(\nabla \mathbf{u}(x))] = 0, \quad \text{in } \ \Omega.$$

in the sense of (3.25). This completes the proof. □

4. Local L^1-Estimate and the Integral Representation of the Pressure

In this section we establish an explicit representation of the pressure on the deformed domain $\mathbf{u}(\Omega)$ in terms of Calderón-Zygmund type singular integral operator of the energy $|\nabla \mathbf{u}|^2$. Our main ideas in the proof are to localize the mollified pressure on the deformed domain $\mathbf{u}(\Omega)$, finding its explicit representation using Green's function of the unit disc in $\mathbb{R}^2$ and finding an uniform estimate by using Calderón-Zygmund Theorem [CZ 52] for $L \log^+ L$ functions.

Proof of Theorem 1.2. Let us assume that $\mathbf{u} \in \mathcal{A}$ minimizes the energy $E[\cdot]$ and $|\nabla \mathbf{u}|^2 \in L \log^+ L$. Let $V \subset\subset \mathbf{u}(\Omega)$ be a smooth and simply connected sub-domain of $\mathbf{u}(\Omega)$. Without loss of generality let us assume that $0 \in V$ and $V = B_1 := \{z \in \mathbb{R}^2 : |z| < 1\}$ be the unit disc. Recall the family $(\mathbf{g}_\varepsilon)_{\varepsilon > 0}$ defined by (3.4) and the family $(q_\varepsilon)_{\varepsilon>0}$ in (3.6) represented by

$$-\nabla q_\varepsilon = \mathbf{g}_\varepsilon, \tag{4.1}$$

modulo an additive constant. Applying the divergence operator to the both sides of the above equation, we obtain

$$-\Delta q_\varepsilon = \mathrm{div}\, \mathbf{g}_\varepsilon. \tag{4.2}$$

Now our idea is to localize the equation (4.2) and find appropriate uniform estimates for the localized q_ε. Let $\eta \in C^\infty_0(B_1)$, $0 \leq \eta \leq 1$ be a cut-off function such that $\eta \equiv 1$ in $B_{2/3}$. Let $\bar{q}_\varepsilon = \eta q_\varepsilon$ be the localized pressure. Then $\bar{q}_\varepsilon$ is the solution to the Dirichlet problem

$$\begin{cases} -\Delta \bar{q}_\varepsilon = f_\varepsilon & \text{in } B_1 \\ \bar{q}_\varepsilon = 0 & \text{on } \partial B_1, \end{cases} \tag{4.3}$$

where $f_\varepsilon := \eta\Delta q_\varepsilon + 2\langle\nabla q_\varepsilon, \nabla\eta\rangle + q_\varepsilon\Delta\eta$. Therefore $\bar{q}_\varepsilon$ is the Green's potential of f_ε in B_1. In other words,

$$\bar{q}_\varepsilon(y) = \int_{B_1} G(z-y) f_\varepsilon(z)\, dz\,, \tag{4.4}$$

where $G(z,y)$ Green's function of the unit disc $B_1 \subset \mathbb{R}^2$ given by

$$G(z,y) := -\frac{1}{2\pi}\log|z-y| + \frac{1}{2\pi}\log(|y||z-\hat{y}|),\ \ \hat{y} := \frac{y}{|y|^2}\,. \tag{4.5}$$

Using (4.1), (4.2) and (4.5) in (4.4), we obtain

$$\begin{aligned}
\bar{q}_\varepsilon(y) \;&=\; -\frac{1}{2\pi}\int_{B_1}\Big(\eta\,\Delta q_\varepsilon + 2\,\langle\nabla q_\varepsilon, \nabla\eta\rangle + q_\varepsilon\,\Delta\eta\Big)\log|z-y|\,dz \\
&\quad+\frac{1}{2\pi}\int_{B_1} f_\varepsilon(z)\log(|y||z-\hat{y}|)\,dz \\
&= \frac{1}{2\pi}\int_{B_1}\Big(\eta\operatorname{div}\mathbf{g}_\varepsilon + 2\,\langle\mathbf{g}_\varepsilon(z), \nabla\eta(z)\rangle - q_\varepsilon\Delta\eta\Big)\,\log|z-y|\,\,dz \\
&\quad+\frac{1}{2\pi}\int_{B_1} f_\varepsilon(z)\log(|y||z-\hat{y}|)\,dz \\
&= \frac{1}{2\pi}\,I^1_\varepsilon(y) + \frac{1}{\pi}\,I^2_\varepsilon(y) + \frac{1}{2\pi}\,I^3_\varepsilon(y) + \frac{1}{2\pi}\,I^4_\varepsilon(y)
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
I^1_\varepsilon(y) \;&:=\; \int_{B_1}\eta(z)\,\log|z-y|\,\operatorname{div}\mathbf{g}_\varepsilon(z)\,dz \\
I^2_\varepsilon(y) \;&:=\; \int_{B_1}\langle\mathbf{g}_\varepsilon(z), \nabla\eta(z)\rangle\,\log|z-y|\,dz \\
I^3_\varepsilon(y) \;&:=\; -\int_{B_1} q_\varepsilon(z)\Delta\eta(z)\log|z-y|\,dz \\
I^4_\varepsilon(y) \;&:=\; \int_{B_1} f_\varepsilon(z)\,\log|y|(|z-\hat{y}|)\,\,dz\,.
\end{aligned}$$

We now establish an uniform local L^1-estimate for q_ε through the following steps.

Step 1: Limits of I^3_ε and I^4_ε Let us fix $|y| < 1/2$. Since $\Delta\eta = 0$ for $|z| < 2/3$, both the integrals $I^3_\varepsilon(y)$ and $I^4_\varepsilon(y)$ are well defined for $|y| < 1/2$. Since q_ε is determined up to a constant, we can add a constant to $z \mapsto \Delta\eta(z)\log z - y|$, if nessecary, to ensure that it has vanishing integral. For each fixed $|y| < 1/2$, let $\mathbf{v}_y : B_1 \to \mathbb{R}^2$ be the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}\mathbf{v}_y(z) = \Delta\eta(z)\log|z-y| & \text{for } z \in B_1 \\ \mathbf{v}_y = 0 & \text{on } \ \partial B_1\,. \end{cases} \tag{4.7}$$

Then using (4.7) and (3.13) we obtain

$$
\begin{aligned}
I^3_\varepsilon(y) &= -\int_{B_1} q_\varepsilon(z)\Delta\eta(z)\log|z-y|\,dz \\
&= -\int_{B_1} q_\varepsilon(z)\operatorname{div}\mathbf{v}_y(z)\,dz \\
&= \int_{B_1} \langle\nabla q_\varepsilon(z),\mathbf{v}_y(z)\rangle\,dz \\
&= -\int_{B_1} \langle\mathbf{g}_\varepsilon(z),\mathbf{v}_y(z)\rangle\,dz \\
&\to -\int_{B_1} \sigma(\mathbf{u}^{-1}(z)):\nabla\mathbf{v}_y(z)\,dz \quad \text{as } \varepsilon\to 0 \\
&:= I^3_0(y)\,.
\end{aligned} \tag{4.8}
$$

Since $f_\varepsilon = \Delta(q_\varepsilon\eta)$ and for each fixed $|y|<1/2$ the function $z\mapsto\Delta\log(|y|(z-\hat{y}))$ is smooth on B_1. By taking $\mathbf{w}_y : B_1\to\mathbb{R}^2$ to be the solution of the Dirichlet problem

$$
\begin{cases}
\operatorname{div}\mathbf{w}_y(z) = \eta(z)\Delta\log(|y||z-\hat{y}| & \text{for } z\in B_1\\
\mathbf{w}_y = 0 & \text{on } \partial B_1,
\end{cases} \tag{4.9}
$$

and applying the above arguments we obtain

$$
\begin{aligned}
I^4_\varepsilon(y) &= \int_{B_1} \Delta\Big(q_\varepsilon(z)\eta(z)\Big)\,\log|y|(|z-\hat{y}|)\,dz \\
&= \int_{B_1} q_\varepsilon(z)\left(\eta(z)\Delta\log(|y||z-\hat{y}|)\right)dz \\
&\to \int_{B_1} \sigma(\mathbf{u}^{-1}(z)):\nabla\mathbf{w}_y(z)\,dz \quad \text{as } \varepsilon\to 0 \\
&:= I^4_0(y)\,.
\end{aligned} \tag{4.10}
$$

Step 2: Limit of I^2_ε Since $\nabla\eta(z)=0$ for $|z|<2/3$, the integral $I^3_\varepsilon(y)$ is well-defined for $|y|<1/2$. Recall that from (3.8) and (3.9)

$$
\begin{aligned}
-g^i_\varepsilon(z) &= \sum_{j=1}^2 \frac{\partial}{\partial z_j}\int_\Omega \sigma_{ij}(x)\,\rho_\varepsilon(\mathbf{u}(x)-z)\,dx \\
&= \sum_{j=1}^2 \frac{\partial}{\partial z_j}\int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(y))\,\rho_\varepsilon(\mathbf{y}-z)\,dy \\
&= \sum_{j=1}^2 \frac{\partial}{\partial z_j}\Big((\sigma_{ij}\circ\mathbf{u}^{-1})*\rho_\varepsilon\Big)(z)\,.
\end{aligned}
$$

In other words,

$$
\mathbf{g}_\varepsilon = -\operatorname{div}\tilde{\sigma}_\varepsilon\,, \tag{4.11}
$$

where the divergence is taken in each rows of matrix $\tilde{\sigma}_\varepsilon := \Big((\tilde{\sigma}_{ij})_\varepsilon\Big)\in\mathbb{M}^{2\times2}$, $(\tilde{\sigma}_{ij})_\varepsilon := (\sigma_{ij}\circ\mathbf{u}^{-1})*\rho_\varepsilon$. Notice that $(\tilde{\sigma}_{ij})_\varepsilon\to\tilde{\sigma}_{ij}:=\sigma_{ij}\circ\mathbf{u}^{-1}$ in L^1 as $\varepsilon\to0$ for each $i,j=1,2$.

Using the above representation of $\mathbf{g}_\varepsilon$ observe that

$$
\begin{aligned}
(4.12)\quad I_\varepsilon^2(y) &= -\int_{B_1} \Big\langle \operatorname{div} \tilde\sigma_\varepsilon(z), \log|z-y|\,\nabla\eta(z) \Big\rangle\, dz \\
&= \int_{B_1} \tilde\sigma_\varepsilon(z) : \nabla\Big(\log|z-y|\,\nabla\eta(z)\Big) dz \\
&= \int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \left(\log|z-y|\,\nabla^2\eta(z) + \frac{\nabla\eta\otimes(z-y)}{|y-z|^2}\right) dz \\
&\to \int_{B_1\setminus B_{2/3}} \tilde\sigma(z) : \left(\log|z-y|\,\nabla^2\eta(z) + \frac{\nabla\eta\otimes(z-y)}{|y-z|^2}\right) dz \quad \text{as } \varepsilon\to 0 \\
&:= I_0^2(y)\,.
\end{aligned}
$$

Step 3: Limit of $I_\varepsilon^1(y)$ Since we assumed $|\nabla\mathbf{u}|^2 \in \mathcal{H}^1_{\text{loc}}(\Omega)$, from the definition of $\tilde\sigma_{ij}$ it follows that $\tilde\sigma_{ij} \in L\log^+ L$. Thus the mollification $(\tilde\sigma_{ij})_\varepsilon$ converges strongly to $\tilde\sigma_{ij}$ in $L\log^+ L$ as $\varepsilon\to 0$. Integrating by parts twice and using (4.11)

$$
\begin{aligned}
I_\varepsilon^1(y) &= \int_{B_1} \operatorname{div}\mathbf{g}_\varepsilon(z)\; \eta(z)\log|z-y|\,dz \\
&= -\int_{B_1} \tilde\sigma_\varepsilon(z) : \nabla^2(\eta(z)\log|z-y|)\,dz \\
&= -\int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \left(\log|z-y|)\,\nabla^2\eta(z) + 2\frac{\nabla\eta(z)\otimes(z-y)}{|z-y|^2}\right) dz \\
&\quad -\int_{B_1} \tilde\sigma_\varepsilon(z) : \left(Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\right) \frac{\eta(z)}{|z-y|^2}\,dz \\
&:= I_\varepsilon^{11}(y) + I_\varepsilon^{12}(y),
\end{aligned}
$$

where Id is the 2×2 identity matrix and

$$
\begin{aligned}
(4.13)\quad I_\varepsilon^{11}(y) &:= -\int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \left(\log|z-y|\,\nabla^2\eta + 2\frac{\nabla\eta\otimes(z-y)}{|z-y|^2}\right) dz \\
&\to -\int_{B_1} \tilde\sigma(z) : \left(\log|z-y|\,\nabla^2\eta + 2\frac{\nabla\eta\otimes(z-y)}{|z-y|^2}\right) dz \quad \text{as } \varepsilon\to 0 \\
&:= I_0^{11}(y)\,,
\end{aligned}
$$

and

$$
(4.14)\qquad I_\varepsilon^{12}(y) := -\int_{B_1} \tilde\sigma_\varepsilon(z) : \left(Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\right) \frac{\eta(z)}{|z-y|^2}\,dz
$$

is the sum of Calderón-Zygmund [CZ 52] type singular integrals with the homogeneous kernel

$$
(4.15)\qquad G_{ij}(z) := \delta_{ij} - 2\frac{z_i z_j}{|z|^2}, \quad z\in\mathbb{R}^2\setminus\{0\}, \quad i,j = 1,2\,.
$$

Observe that each G_{ij} satisfies all the conditions of the Calderón-Zygmund Theorem 1.3 [CZ 52]. Since $\sigma_{ij} \in L\log^+ L$, the following sum of singular integrals

$$
(4.16)\qquad I_0^{12}(y) := -\int_{B_1} \tilde\sigma(z) : \left(Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\right) \frac{\eta(z)}{|z-y|^2}\,dz
$$

exists for almost every $|y| < 1/2$ and is integrable.

Claim: $I^{12}_\varepsilon \to I^{12}_0$ *strongly in* $L^1(B_{1/2})$.

Proof. Let $\rho > 1/2$ and extend $\tilde\sigma_{ij}$ by 0 outside the unit ball B_1. From the singular integrals (4.14) and (4.16), we have

$$I^{12}_\varepsilon(y) - I^{12}_0(y) = -\sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta \left((\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij} \right) \left(\delta_{ij} - 2\frac{(z_i - y_i)(z_j - y_j)}{|z-y|^2} \right) \frac{dz}{|z-y|^2} .$$

Extend I^{12}_ε and I^{12}_0 by 0 outside the ball $B_{1/2}$. Then by using Calderón-Zygmund estimate in Theorem 1.3 and strong convergence of $(\tilde\sigma_{ij})_\varepsilon$ in $L\log^+ L$, for any $\rho > 1/2$ we obtain

$$\begin{aligned}
\int_{B_{1/2}} |I^{12}_\varepsilon(y) - I^{12}_0(y)| dy &= \int_{B_\rho} |I^{12}_\varepsilon(y) - I^{12}_0(y)| dy \\
&\le C \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \, dz + C(\operatorname{meas} B_\rho)^{-\frac{1}{2}} \\
&\quad + C \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \, \log^+ \left((\operatorname{meas} B_\rho)^{\frac{3}{2}} \eta |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \right) dz \\
&\le C(1 + \log^+ \rho) \sum_{i,j=1}^{2} \int_{B_1} \eta |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \, dz + \frac{C}{\rho} \\
&\quad + C \sum_{i,j=1}^{2} \int_{B_1} |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \, \log \left(2 + |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \right) dz \\
&\to \frac{C}{\rho} \quad \text{as } \varepsilon \to 0 \\
&\to 0 \quad \text{as } \rho \to \infty .
\end{aligned}$$

Hence $I^{12}_\varepsilon \to I^{12}_0$ strongly in $L^1(B_{1/2})$. This proves the claim. $\square$

Step 4: An explicit representation of the pressure To complete the proof, let us define $q : B_{1/2} \to \mathbb{R}$ by

$$q(y) := \frac{1}{2\pi} \left(I^{11}_0(y) + I^{12}_0(y) \right) + \frac{1}{\pi} I^2_0(y) + \frac{1}{2\pi} \left(I^3_0(y) + I^4_0(y) \right) .$$

Then from (4.9), (4.11), (4.12), (4.13) and (4.16), we conclude that $q_\varepsilon \to q$ strongly in $L^1(B_{1/2})$ and q is represented as

$$\begin{aligned}
(4.17) \quad q(y) &= \frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left(\nabla_z (\mathbf{w}_y(z) - \mathbf{v}_y(z)) - 2\Delta\eta \log|z-y| \right) dz \\
&\quad - \frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left(Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2} \right) \frac{\eta(z)}{|z-y|^2} \, dz,
\end{aligned}$$

where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2\times 2}$ given by the equation (3.9). Since q is the strong limit of the family q_ε in ball $B_{1/2}$, it is independent of the choice of the cut-off function η. Following the same arguments as in Section 3, we can extend q to all of $\mathbf{u}(\Omega)$ such that $q \in L^1_{\text{loc}}(\mathbf{u}(\Omega))$ and the pair $(\mathbf{u}, q)$ satisfies the identity (1.7). This completes the proof of Theorem 1.2. $\square$

5. Partial Regularity

Let us denote $L(x, \nabla \mathbf{u}) = \nabla \mathbf{u} - p(x)\nabla \mathbf{u}^{-t}$, then the equation is $\operatorname{div} L(x, \nabla \mathbf{u}) = 0$. First let us examine the ellipticity condition $L_{ij}(x,\xi)\xi_{ij} \geq \lambda|\xi|^2$ for some $\lambda > 0$. Since the deformation is incompressible we obtain

$$\nabla \mathbf{u}^{-t} = \begin{pmatrix} u^2_2 & -u^2_1 \\ -u^1_2 & u^1_1 \end{pmatrix}. \tag{5.1}$$

Introduce $I = L_{ij}(x,\xi)\xi_{ij} = |\xi|^2 - 2p(x)\det \xi$, where ξ is any 2×2 matrix. Then completing squares we get

$$\begin{aligned} I &= \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2 - 2p(\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) \\ &= (\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (1-p^2)(\xi_{22}^2 + \xi_{21}^2) \\ &= (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1-p^2)(\xi_{11}^2 + \xi_{12}^2). \end{aligned} \tag{5.2}$$

Adding both identities and dividing by 2 we arrive at

$$\begin{aligned} I &= \frac{1}{2}((\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1-p^2)|\xi|^2 \\ &\geq \frac{1-p^2}{2}|\xi|^2. \end{aligned}$$

This computation shows that ellipticity condition

$$L_{ij}(x,\xi)\xi_{ij} \geq \lambda|\xi|^2, \lambda > 0$$

is equivalent to assume that

$$p^2 \leq 1 - 2\lambda. \tag{5.3}$$

Note that p is defined up to addition of arbitrary constant, thus (5.3) is satisfied in subdomain $D \subset \Omega$ if

$$\operatorname{osc}_D p^2 < 1.$$

Next we examine the strong ellipticity condition, i.e.

$$L_{ij,kl}(x,\eta)\xi_{ij}\xi_{kl} \geq \lambda|\xi|^2, \tag{5.4}$$

where η stands as dummy variable for $\nabla \mathbf{u}$. Recall that

$$L_{ij}(x,\eta) = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} - p(x) \begin{pmatrix} \eta_{22} & -\eta_{21} \\ -\eta_{12} & \eta_{11} \end{pmatrix}. \tag{5.5}$$

For instance $L_{11,kl} = \delta_{11,kl} - p\delta_{22,kl}$, and it is easy to check that

$$L_{ij,kl}(x,\eta)\xi_{ij}\xi_{kl} = |\xi|^2 - 2p(x)\det \xi,$$

that is the ellipticity implies strong ellipticity.

In what follows we make the following two assumptions

1 $\mathbf{u}$ is $W^{1,3}(\Omega)$

2 $q(z)$ is α-Hölder continuous with respect to z.

Proposition 5.1. *Under the assumptions* **1-2** *we have that*

(i)
$$|L_{ij}(x, \nabla \boldsymbol{u})| \le L(1 + |\nabla \boldsymbol{u}|)$$

(ii) *for any* $x_1, x_2 \in \overline{\Omega}, \eta \in M^{2\times 2}$
$$\frac{|L_{ij}(x_1, \eta) - L_{ij}(x_2, \eta)|}{1 + |\eta|} \le C|x_1 - x_2|^\alpha$$

(iii) L_{ij} *is differentiable with respect to* η *with bounded and continuous derivatives*
$$|L_{ij,kl}(x, \eta)| \le L$$

(iv) L_{ij} *satisfies to strong ellipticity condition*
$$L_{ij,kl}(x, \eta)\eta_{ij}\eta_{kl} \ge \lambda|\eta|^2$$

Proof: Since $\mathbf{u} \in W^{1,3}$, Sobolev imbedding theorem implies that $\mathbf{u} \in C^{1/3}$ then $p(x) = q(\mathbf{u}(x))$ is Hölder continuous and (i)-(ii) follow. (iii)-(iv) follow from (5.3). $\square$

Remark 5.2. Assumptions $(i) - (iv)$ are stated in [GM 79], if fact they consider more general systems of elliptic equations. Using their theorem 1 we can obtain the following partial regularity result.

Theorem 5.3. *Assume that assumptions* **1-2** *are satisfied, i.e.* $\boldsymbol{u} \in W^{1,3}(\Omega), q \in C^\alpha$. *Then the first derivatives of* $\boldsymbol{u}$ *are Hölder continuous on an open set* Ω_0. *Moreover*
$$|\Omega \setminus \Omega_0| = 0.$$

Proof: It follows from proposition then the requirements of theorem 1 in [GM 79] are satisfied and the result follows. $\square$

Acknowledgement This work was initiated while both authors were at the Australian National University, which was supported by the Australian Research Council. The second author was partially supported by the National Science Foundation Grant DMS - 0140338.

References

[Ad 75] Adams,R.: *Sobolev Spaces.* Academic Press, New York, 1975.

[Ba 77] Ball,J.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.* **64**, 337–403 (1977).

[BOP 91] Bauman,P., Owen,N.C., Phillips,D.: Maximal smoothness of solutions to certain Euler-Lagrange equations from nonlinear elasticity. *Proc. Roy. Soc. Edinburgh Sect. A* **119**, 241–263 (1991).

[BOP 91a] Bauman,P., Owen,N.C., Phillips,D.: Maximum principles and a priori estimates for a class of problems from nonlinear elasticity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8**, 119–157 (1991).

[BOP 92] Bauman,P., Owen,N.C., Phillips,D.: Maximum principles and an a priori estimates for an incompressible material in nonlinear elasticity. *Comm. Partial Differential Equations* **17**, 1185-1212 (1992).

[CZ 52] Calderon,A.P., Zygmund,A.: On the existence of certain singular integrals. *Acta Math.* **88**, 85–139 (1952).

[CHK 08] Chaudhuri,N., Hakobyan,A., Karakhanyan,A.L.: Forthcoming.

[DM 90] Dacorogna,B., Moser,J,: On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Linairé* **7**, 1–26 (1990).

[Ev 98] Evans,L.C.: *Partial Differential Equations*. Graduate Studies in Mathematics, **19**, American Mathematical Society, 1998.

[EG 99] Evans,L.C., Gariepy,R.F,: On the partial regularity of energy-minimizing, area-preserving maps, *Calc. Var. Partial Differential Equations* **9**, 357-372 (1999).

[Ga 94] Galdi,G.P.: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* Volume I, Springer Tracts in Natural Philisophy **38**, Springer-Verlag, 1994.

[Gi 83] Giaquinta,M.: *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Annals of Mathematics Studies **105**, Princeton University Press, Princeton, NJ, 1983.

[GM 79] Giaguinta,M., Modica,G.: *Almost-everywhere regularity results for solutions of nonlinear elliptic systems.* Manuscripta Math. **28**, 109-158 (1979).

[GT 97] Gilbarg,D. Trudinger,N.S.: *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[IS 93] Iwaniec,T., Šverak,V.: On mappings with integrable dilatiation. *Proc. Amer. Math. Soc* **118**, 181–188 (1993).

[Mo 52] Morrey,C.B.: Quasiconvexity and the semicontinuity of multiple integrals. *Pacific. J. Math.* **2**, 25–52 (1952).

[Og 84] Ogden,R.W.: *Non-linear elastic deformations.* Ellis Horwood Ltd. Chichester, 1984.

[St 69] Stein,E.: Note on the class L log L, *Studia Math.* **32**, 305-301 (1969).

[St 70] Stein,E.: *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, NJ, 1970.

[Sv 88] Šverák,V.: Regularity properties of deformations with finite energy. *Arch. Rat. Mech. Anal.* **100**, 105–127 (1988).

[TO 81] Tallec,P.L., Oden,J.T.: Existence and characterization of hydrostatic pressure in finite deformations of incompressible elastic bodies. *J. Elasticity* **11**, 341–357 (1981).

School of Mathematics and Applied Statistics University of Wollongong, Wollongong, NSW 2522, Australia
E-mail address: chaudhur@uow.edu.au

Department of Mathematics, University of Texas at Austin,Austin, Texas 78712, USA
E-mail address: aram@math.utexas.edu